At the point of intersection of the two orbits, $\rho = r$, and so by (9),

$$\cos\frac{\theta}{2} = \sqrt{\frac{a}{r}}, \qquad \sin\frac{\theta}{2} = \sqrt{\frac{r-a}{r}}, \qquad \text{and} \qquad \tan\frac{\theta}{2} = \sqrt{\frac{r-a}{a}},$$

(10) then becoming

$$2t = 2a^{3/2}\sqrt{\frac{2}{\mu}}\left(\sqrt{\frac{r-a}{a}} + \frac{1}{3}\sqrt{\frac{(r-a)^3}{a^3}}\right) = \frac{2}{3}\sqrt{\frac{2}{\mu}}(r+2a)\sqrt{r-a}, \tag{11}$$

which by the problem is to be a maximum. For this, the first derivative of both members of (11) with respect to α must vanish, giving $r = 2a$.

$$\text{Hence, } \theta = \frac{\pi}{2}, \text{ and the time, } \frac{\theta r}{v} = \frac{2\pi a\sqrt{2a}}{\sqrt{\mu}};$$

and (11) becomes

$$t = \frac{4a}{3}\sqrt{\frac{2a}{\mu}},$$

and the greatest part of the earth's year during which a parabolic comet can remain in the earth's orbit is given by

$$\frac{4a}{3}\sqrt{\frac{2a}{\mu}} \div \frac{2\pi a\sqrt{2a}}{\sqrt{\mu}} = \frac{2}{3\pi}.$$

Again by the theory of central forces, in the notation above defined, the velocity v_1 of the comet is

$$v_1 = \frac{h}{p}, \text{ or, by (1) and (5), } v_1^2 = \frac{2a\mu}{a\rho} = \frac{2\mu}{\rho}.$$

This value of v_1 is greatest when ρ is least, which is at perihelion when $\rho = a$; so that the required maximum velocity is

$$v_1 = \sqrt{\frac{2\mu}{a}}.$$

Also solved by C. F. GUMMER and O. S. ADAMS.

NUMBER THEORY.

248. Proposed by E. T. BELL, Seattle, Washington.

NOTE BY THE PROPOSER.

The solution in the June MONTHLY (p. 295) is obviously incomplete. It is not shown that: (1) for general n the expression under the radical is a perfect square; (2) the stated form of Δ_n is both necessary and sufficient. The completion of (1) will prove only the sufficiency; the necessity is also asked for in the problem.

256. Proposed by FRANK IRWIN, University of California.

Let p be an odd prime, and let the notation $1/k$ stand for the solution of $kx \equiv 1 \pmod p$. Then show that if the sum of the numbers $1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{(p-1)/2}$ be congruent to zero (mod p), should that be possible, the same is true for the sum of their products two at a time, as well as four at a time.

SOLUTION BY E. B. ESCOTT, Kansas City, Mo.

Since the congruence $(x-1)(2x-1)(3x-1)\cdots[(p-1)x-1] \equiv 0 \pmod p$ has $p-1$ roots, namely, $1, 2, 3, \cdots (p-1)$, and since the same is true of the congruence $1 - x^{p-1} \equiv 0$ (mod p), by Fermat's Theorem it follows that

$$(x-1)(2x-1)(3x-1)\cdots[(p-1)x-1] \equiv 1 - x^{p-1} \tag{1}$$

is an identical congruence. It is also satisfied by $x \equiv 0$ so that it has p roots while it is only of the $(p-1)$th degree.

The last may be written

$$(x-1)(2x-1)\cdots[\tfrac{1}{2}(p-1)x-1][-\tfrac{1}{2}(p-1)x-1]\cdots(-2x-1)(-x-1) \equiv 0,$$

or

$$(x-1)(2x-1)\cdots[\tfrac{1}{2}(p-1)x-1]\cdot(-1)^{1/2(p-1)}(x+1)(2x+1)\cdots[\tfrac{1}{2}(p-1)x+1] \equiv 0;$$

whence

$$(x-1)(x-\tfrac{1}{2})\cdots\left(x-\frac{1}{\tfrac{1}{2}(p-1)}\right)\cdot(x+1)(x+\tfrac{1}{2})\cdots\left(x+\frac{1}{\tfrac{1}{2}(p-1)}\right) \equiv 0.$$

Let

$$S_1 = 1 + \tfrac{1}{2} + \cdots + \frac{1}{\tfrac{1}{2}(p-1)},$$

$$S_2 = 1\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{3} + \cdots, \quad \text{etc.}$$

The congruence may be written

$$(x^{1/2(p-1)} - S_1x^{1/2(p-1)-1} + S_2x^{1/2(p-1)-2} - \cdots)\cdot(x^{1/2(p-1)} + S_1x^{1/2(p-1)-1} + S_2x^{1/2(p-1)-2} + \cdots) \equiv 0,$$

i. e.,

$$(x^{1/2(p-1)} + S_2x^{1/2(p-1)-2} + \cdots)^2 - (S_1x^{1/2(p-1)-1} + \cdots)^2 \equiv 0.$$

Expanding, we find the coefficients of the first few powers of x to be 1, $2S_2 - S_1^2$, $S_2^2 + 2S_4 - 2S_1S_3$, $\cdots$. Since by (1) this congruence is identical with $1 - x^{p-1}$, it follows that

$$2S_2 - S_1^2 \equiv 0 \pmod{p}, \tag{2}$$

and

$$S_2^2 + 2S_4 - 2S_1S_3 \equiv 0 \pmod{p}. \tag{3}$$

Therefore, by (2) if $S_1 \equiv 0$, it follows that $S_2 \equiv 0$ and by (3) that $S_4 \equiv 0$, which was to be proved.

NOTE: The same property is true of any $\frac{1}{2}(p-1)$ numbers, where the sum of no two is congruent to zero (mod p). The proof is the same.

Examples: (1) If $p = 7$;

$$1 + 2 + 4 = 7 \equiv 0 \pmod{7},$$

$$1\cdot 2 + 1\cdot 4 + 2\cdot 4 = 14 \equiv 0 \pmod{7}.$$

(2) If $p = 11$;

$$1 + 3 + 4 + 5 + 9 = 22 \equiv 0 \pmod{11},$$

$$1\cdot 3 + 1\cdot 4 + \cdots = 176 \equiv 0 \pmod{11},$$

$$1\cdot 3\cdot 4\cdot 5 + 1\cdot 3\cdot 4\cdot 9 + \cdots = 1{,}023 \equiv 0 \pmod{11}.$$

Also solved by C. F. GUMMER.